Jeans analysis of self-gravitating systems in f(R)- gravity

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Outlines

- Dust- dominated self-gravitating systems
- The Newtonian limit of f(R)-gravity
- Jeans criterion for gravitational instability in f(R)-gravity
- The Jeans mass limit in f(R)-gravity
- Discussion and conclusions
- Next steps

Dust- dominated self-gravitating systems

The collapse of self-gravitational collisionless systems can be dealt with the introduction of coupled collisionless Boltzmann and Poisson equations

$$\frac{\partial f(\vec{r},\vec{v},t)}{\partial t} + (\vec{v}\cdot\vec{\nabla}_r) f(\vec{r},\vec{v},t) - (\vec{\nabla}\Phi\cdot\vec{\nabla}_v) f(\underbrace{\vec{r},\vec{v}},t) = 0$$

three-dimensional vectors in the spatial manifold

$$\vec{\nabla}^2\Phi(\vec{r},t) = 4\pi G \int f(\vec{r},\vec{v},t)\, d\vec{v},$$

A self-gravitating system at equilibrium is described by a time-independent distribution function $f_o(x, v)$ *and a potential* $\Phi_0(x)$ *that are solutions of above equations*

J. Binney and S. Tremaine, Galactic Dynamics (Princeton University Press, Princeton, NJ, 1994).

Dust- dominated self-gravitating systems

Considering a small perturbation to this equilibrium:

$$f(\vec{r}, \vec{v}, t) = f_0(\vec{r}, \vec{v}) + \epsilon f_1(\vec{r}, \vec{v}, t),$$

$$\Phi(\vec{r}, t) = \Phi_0(\vec{r}) + \epsilon \Phi_1(\vec{r}, t),$$

★ Where $\varepsilon << 1$ and

by substituting in Boltzmann and Poisson equations and by linearizing, one obtains:

$$\frac{\partial f_1(\vec{r}, \vec{v}, t)}{\partial t} + \vec{v} \cdot \frac{\partial f_1(\vec{r}, \vec{v}, t)}{\partial \vec{r}} - \vec{\nabla}\Phi_1(\vec{r}, t) \cdot \frac{\partial f_0(\vec{r}, \vec{v})}{\partial \vec{v}} - \vec{\nabla}\Phi_0(\vec{r}) \cdot \frac{\partial f_1(\vec{r}, \vec{v}, t)}{\partial \vec{v}} = 0,$$

$$\vec{\nabla}^2 \Phi_1(\vec{r}, t) = 4\pi G \int f_1(\vec{r}, \vec{v}, t) d\vec{v}.$$

Dust- dominated self-gravitating systems

Since the equilibrium state is assumed to be homogeneous and time-independent, one can set $f_o(x,v,t) = f(v)$, and the so-called **Jeans "swindle" to set $\Phi_o = 0$**

In Fourier components

$$-i\omega f_1 + \vec{v}\cdot(i\vec{k}f_1) - (i\vec{k}\Phi_1)\cdot\frac{\partial f_0}{\partial \vec{v}} = 0,$$

$$-k^2\Phi_1 = 4\pi G\int f_1 d\vec{v}.$$

By combining these equations, we obtain the **dispersion relation**

$$1 + \frac{4\pi G}{k^2}\int\frac{\vec{k}\cdot\frac{\partial f_0}{\partial \vec{v}}}{\vec{v}\cdot\vec{k}-\omega d\vec{v}} = 0$$

Dust- dominated self-gravitating systems

In the case of stellar systems, by assuming a Maxwellian distribution function for f_o we have

$$f_0 = \frac{\rho_0}{(2\pi\sigma^2)^{(3/2)}} e^{-(v^2/2\sigma^2)}$$

$$1 - \frac{2\sqrt{2\pi}G\rho_0}{k\sigma^3} \int \frac{v_x e^{-(v_x^2/2\sigma^2)}}{kv_x - \omega} dv_x = 0.$$

By setting $\omega = 0$, the limit for instability is obtained: $k^2(\omega = 0) = \frac{4\pi G\rho_0}{\sigma^2} = k_J^2,$

by which it is possible to define the Jeans mass (M_J) as the mass originally contained within a sphere of diameter λ_J:

$$M_J = \frac{4\pi}{3}\rho_0\left(\frac{1}{2}\lambda_J\right)^3,$$

Where $\lambda_J^2 = \frac{\pi\sigma^2}{G\rho_0}$ is the Jeans length

....and then we can write $M_J = \frac{\pi}{6}\sqrt{\frac{1}{\rho_0}}\cdot\left(\frac{\pi\sigma^2}{G}\right)^3$

Dust- dominated self-gravitating systems

In order to evaluate the integral in the dispersion relation, we have to study the singularity at $\omega = k\, v_x$. To this end, it is useful to write the dispersion relation as

$$1 - \frac{k_J^2}{k^2} W(\beta) = 0,$$

defining $$W(\beta) \equiv \frac{1}{\sqrt{2\pi}} \int \frac{x e^{-(x^2/2)}}{x - \beta} dx,$$

- Where $\beta = \frac{\omega}{k\sigma}$ and $x = \frac{v_x}{\sigma}$
- We set also $\omega = i\omega_I$ and $Re[W(\frac{\omega}{k\sigma})] = 0$ because we are interested in the unstable modes

These modes appear when the imaginary part of ω is greater than zero and in this case the integral in the dispersion relation can be resolved just with previous prescriptions.

Dust- dominated self-gravitating systems

In order to study unstable models we replace the following identities

$$\int_0^\infty \frac{x^2 e^{-x^2}}{x^2+\beta^2}\,dx = \frac{1}{2}\sqrt{\pi} - \frac{1}{2}\pi\beta e^{\beta^2}[1-\mathrm{erf}\,\beta],$$

$$\mathrm{erf}\,\beta(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$$

into the dispersion relation obtaining:

$$k^2 = k_J^2\left\{1 - \frac{\sqrt{\pi}\,\omega_I}{\sqrt{2}k\sigma} e^{(\omega_I/\sqrt{2}k\sigma}\left[1 - \mathrm{erf}\left(\frac{\omega_I}{\sqrt{2}k\sigma}\right)\right]\right\}.$$

This is the standard dispersion relation describing the criterion to collapse for infinite homogeneous fluid and stellar systems

The Newtonian limit of f(R) - gravity

Field equations in f(R)-gravity give rise to the modified Poisson equations. We know that

$$R^{(2)} \simeq \frac{1}{2}\nabla^2 g^{(2)}_{00} - \frac{1}{2}\nabla^2 g^{(2)}_{ii}$$

Also we well konwn that $R^{(2)} \simeq \nabla^2(\Phi - \Psi)$

- Ψ is the further gravitational potential related to the metric component $g^{(2)}_{ii}$

...and then the field equations assume this form

$$\nabla^2\Phi + \nabla^2\Psi - 2f''(0)\nabla^4\Phi + 2f''(0)\nabla^4\Psi = 2\mathcal{X}\rho$$

$$\nabla^2\Phi - \nabla^2\Psi + 3f''(0)\nabla^4\Phi - 3f''(0)\nabla^4\Psi = -\mathcal{X}\rho.$$

S. Capozziello, M. De Laurentis Phys. Rep. 509, 167-321 (2011)

S. Capozziello, M. De Laurentis Ann. Phys. 524, 545 (2012)

Jeans criterion for gravitational instability in f(R)-gravity

Let us assume the standard collisionless Boltzmann equation:

$$\frac{\partial f(\vec{r},\vec{v},t)}{\partial t} + (\vec{v}\cdot\vec{\nabla}_r) f(\vec{r},\vec{v},t) - (\vec{\nabla}\Phi\cdot\vec{\nabla}_v) f(\vec{r},\vec{v},t) = 0,$$

Where, according to the Newtonian theory, only the potentia Φ *is present*

Considering the f(R) Poisson equations, also the potential Ψ *has to be considered so we obtain the coupled equations*

$$\nabla^2(\Phi+\Psi) - 2\alpha\nabla^4(\Phi-\Psi) = 16\pi G\int f(\vec{r},\vec{v},t)d\vec{v}$$

$$\nabla^2(\Phi-\Psi) + 3\alpha\nabla^4(\Phi-\Psi) = -8\pi G\int f(\vec{r},\vec{v},t)d\vec{v}.$$

★ *We have replaced f'(0) with the greek letter* α

Jeans criterion for gravitational instability in f(R)-gravity

As in standard case, we consider small perturbation to the equilibrium and linearize the equations and in Fourier space so they became

$$-i\omega f_1 + \vec{v}\cdot(i\vec{k}f_1) - (i\vec{k}\Phi_1)\cdot\frac{\partial f_0}{\partial \vec{v}} = 0,$$

$$-k^2(\Phi_1 + \Psi_1) - 2\alpha k^4(\Phi_1 - \Psi_1) = 16\pi G\int f_1 d\vec{v},$$

$$k^2(\Phi_1 - \Psi_1) - 3\alpha k^4(\Phi_1 - \Psi_1) = 8\pi G\int f_1 d\vec{v}.$$

Jeans criterion for gravitational instability in f(R)-gravity

Combining the above equations we obtain a relation between Φ_1 *and* Ψ_1

$$\Psi_1 = \frac{3 - 4\alpha k^2}{1 - 4\alpha k^2}\Phi_1$$

And then the dispersion relation is

$$1 - 4\pi G\frac{1 - 4\alpha k^2}{3\alpha k^4 - k^2}\int\left(\frac{\vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}}}{\vec{v}\cdot\vec{k} - \omega}\right)d\vec{v} = 0.$$

As in standard case, one can write

$$1 + \frac{2\sqrt{2\pi}G\rho_0}{\sigma^3}\frac{1 - 4\alpha k^2}{3\alpha k^4 - k^2}\left[\int\frac{kv_x e^{-(v_x^2/2\sigma^2)}}{kv_x - \omega}dv_x\right] = 0.$$

By eliminating the higher-order terms (imposing $\alpha = 0$*), We obtain again the standard dispersion*

Jeans criterion for gravitational instability in f(R)-gravity

In order to compute the integral in the dispersion relation , we consider the same approach used in the classical case, and finally we obtain:

$$1+\mathcal{G}\frac{1-4\alpha k^2}{3\alpha k^4-k^2}\left[1-\sqrt{\pi}\,x e^{x^2}(1-\mathrm{erf}[x])\right]=0,$$

- Where $x=\frac{\omega_I}{\sqrt{2}k\sigma}$ and $\mathcal{G}=\frac{4G\pi\rho_0}{\sigma^2}$

To compare the modified and calssical dispersion relation we to normalize the equation to the classical Jeans length by fixing the parameter of f(R)- gravity, that is

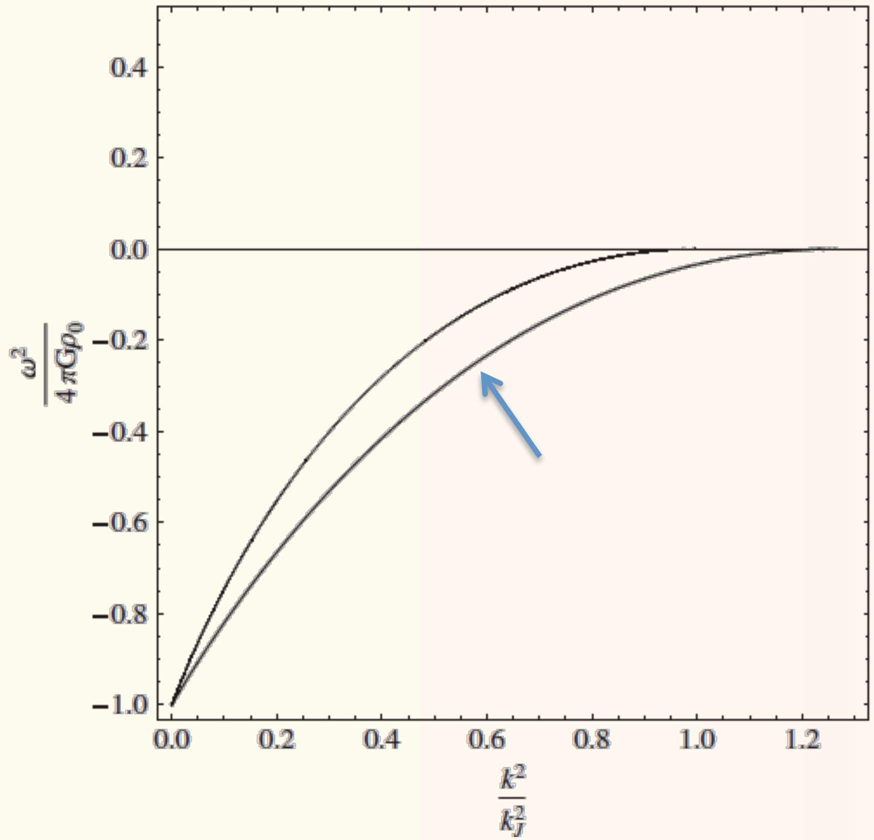
$$\alpha=-\frac{1}{k_j^2}=-\frac{\sigma^2}{4\pi G\rho_0}.$$

This parameterization is correct because the dimension (an inverse of squared length) allows us to parametrize as in standard case

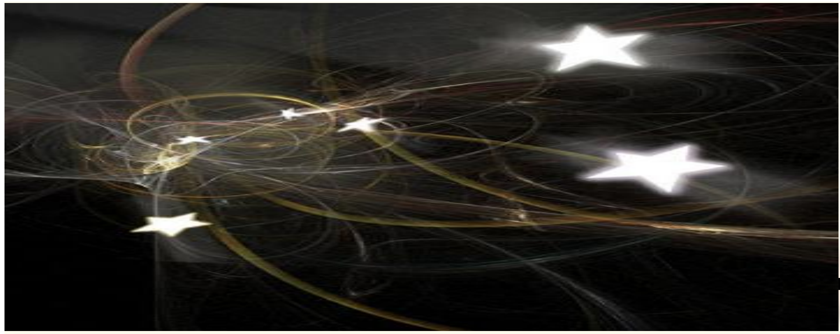
Jeans criterion for gravitational instability in f(R)-gravity

Finally we write and plot this relation

$$\frac{3k^4}{k_j^4}+\frac{k^2}{k_j^2}=\left(\frac{4k^2}{k_j^2}+1\right)[1-\sqrt{\pi}xe^{x^2}(1-\mathrm{erf}[x])]=0.$$



The bold line indicates the plot of the modified dispersion relation.
The thin line indicates the plot of the standard dispersion equation



The Jeans mass limit in f(R)-gravity

A numerical estimation of the f(R) instability length in terms of the standard Newtonian one can be achieved

By solving numerically the above equation with the condition ω = 0, we obtain that the collapse occurs for

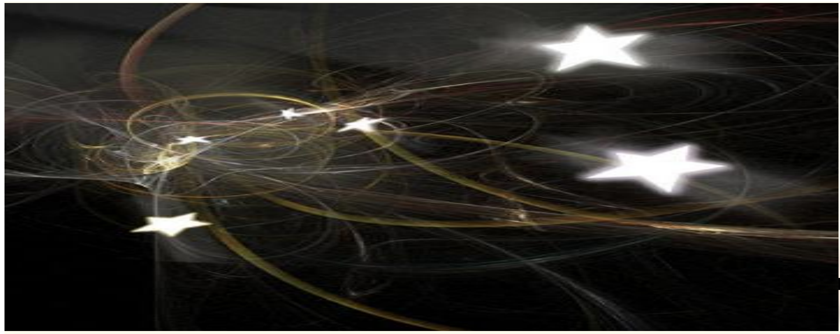
$$k^2 = 1.2637 k_J^2$$

However we can estimate also analytically the limit for the instability

In order to evaluate the Jeans mass limit in f(R)- gravity, we set ω = 0

$$3\sigma^2 \alpha k^4 - (16\pi G\rho_0 \alpha + \sigma^2)k^2 + 4\pi G\rho_0 = 0.$$

The additional condition α < 0 discriminates the class of viable f(R) models: in such a case we obtain stable cosmological solution and positively defined massive states



The Jeans mass limit in f(R)-gravity

This $\alpha<0$ condition selects the physically viable models allowing to solve the above equation for real values of k.

In particular, the above numerical solution can be recast as $$k^2 = \frac{2}{3}(3+\sqrt{21})\pi\frac{G\rho}{\sigma^2}.$$

The relation to the Newtonian value of the Jeans instability is $$k^2 = \tfrac{1}{6}(3+\sqrt{21})k_J^2.$$

Now, we can define the new Jeans mass as $$\tilde{M}_J = 6\sqrt{\frac{6}{(3+\sqrt{21})^3}}M_J$$

which is proportional to the standard Newtonian value

We will confront this specific solutions with some observed structures.

The M_J – T relation

One can deal with the star formation problem in two ways:

- We can take into account the formation of individual stars and
- We can discuss the formation of the whole star system starting from interstellar clouds

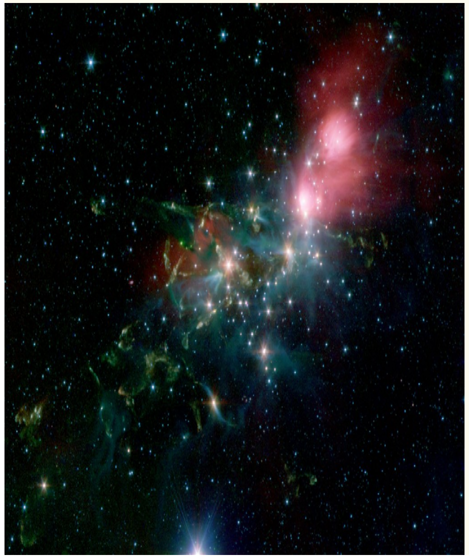
To answer these problems it is very important to study then interstellar medium (ISM) and its properties

The ISM physical conditions in the galaxies change in a very wide range, from hot X-ray emitting plasma to cold molecular gas, so it is very complicated to classify the ISM by its properties

The M_J – T relation

However, we can distinguish, in the first approximation, between

- **Diffuse hydrogen clouds**. The most powerful tool to measure the properties of these clouds is the 21 cm line emission of HI. They are cold clouds so the temperature is in the range $10 \div 50$ K, and their extension is up to $50 \div 100$ kpc from galactic center



- **Diffuse molecular clouds** are generally self-gravitating, magnetized, turbulent fluids systems, observed in sub-mm. The most of the molecular gas is H_2, and the rest is CO. Here, the conditions are very similar to the HI clouds but in this case, the cloud can be more massive. They have, typically, masses in the range $3 \div 100\ M_\odot$, temperature in $15 \div 50$ K and particle density in $(5 \div 50) \times 10^8\ m^{-3}$.

The M_J – T relation

- ***Giant molecular clouds*** *are very large complexes of particles (dust and gas), in which the range of the masses is typically $10^5 \div 10^6 M_\odot$ but they are very cold.*
  The temperature is ≈ 15 K, and the number of particles is $(1 \div 3) \times 10^8$ m^{-3} .
  However, there exist also small molecular clouds with masses $M < 10^4 M_\odot$.
  They are the best sites for star formation, despite the mechanism of formation does not recover the star formation rate that would be $250 M_\odot$ yr^{-1}

The M_J – T relation

- ***HII regions***. They are ISM regions with temperatures in the range $10^3 \div 10^4$ K, emitting primarily in the radio and IR regions. At low frequencies, observations are associated to free-free electron transition (thermaL Bremsstrahlung). Their densities range from over a million particles per cm^3 in the ultracompact H II regions to only a few particles per cm^3 in the largest and most extended regions. This implies total masses between 10^2 and 10^5 $M_\odot$

- ***Bok globules*** are dark clouds of dense cosmic dust and gas in which star formation sometimes takes place. Bok globules are found within H II regions, and typically have a mass of about 2 to 50 $M_\odot$ contained within a region of about a light year.

The M_J – T relation

Using very general conditions, we want to show the difference in the Jeans mass Value between standard and f(R)- gravity.

Let us take into account $$M_J = \frac{\pi}{6}\sqrt{\frac{1}{\rho_0}\left(\frac{\pi\sigma^2}{G}\right)^3},$$

in which ρ_o is the ISM density and σ is the velocity dispersion of particles due to the temperature

These two quantities are defined as $\rho_0 = m_H n_H \mu,$ and $\sigma^2 = \frac{k_B T}{m_H}$

Where $\mathbf{n_H}$ is the number of particles measured in $\mathbf{m^{-3}}$, is the mean molecular Weight, $\mathbf{k_B}$ is the Boltzmann constant and $\mathbf{m_H}$ is the proton mass

By using these relations, we are able to compute the Jeans mass for interstellar clouds and to plot its behavior against the temperature

The M_J – T relation

Any astrophysical system reported in Table is associated to a particular (M_J – T)-region.

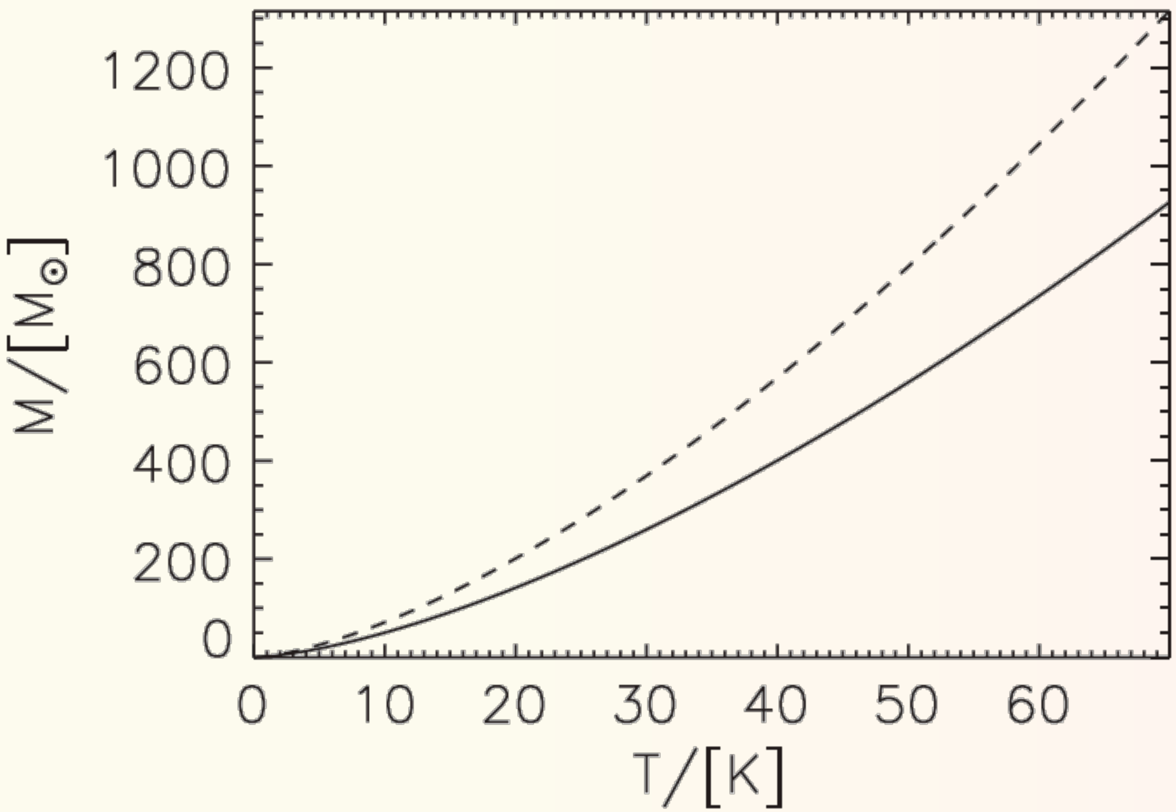
Subject	T (K)	n (10^8 m^{-3})	μ	M_J ($M_\odot$)	$\tilde{M}_J$ ($M_\odot$)
Diffuse hydrogen clouds	50	5.0	1	795.13	559.68
Diffuse molecular clouds	30	50	2	82.63	58.16
Giant molecular clouds	15	1.0	2	206.58	145.41
Bok globules	10	100	2	11.24	7.91

Differences between the two theories for any self-gravitating system are clear

The M_J – T relation

Dashed-line indicates the Newtonian Jeans mass behavior with respect to the temperature.

Continue-line indicates the same for $f(R)$-gravity Jeans mass.



The M_J – T relation

By referring to the catalog of molecular clouds in Roman-Duval et al., Astrophys. J. 723, 492 (2010), we have calculated the Jeans mass in the Newtonian and f(R) cases.

In all cases we note a substantial difference between the classical and f.R) value.

Subject	T K	n (10^8 m^{-3})	M_J ($M_\odot$)	$\tilde{M}_J$ ($M_\odot$)
GRSMC G 053.59 + 00.04	5.97	1.48	18.25	12.85
GRSMC G 049.49 − 00.41	6.48	1.54	21.32	15.00
GRSMC G 018.89 − 00.51	6.61	1.58	22.65	15.94
GRSMC G 030.49 − 00.36	7.05	1.66	22.81	16.06
GRSMC G 035.14 − 00.76	7.11	1.89	28.88	20.33
GRSMC G 034.24 + 00.14	7.15	2.04	29.61	20.84
GRSMC G 019.94 − 00.81	7.17	2.43	29.80	20.98
GRSMC G 038.94 − 00.46	7.35	2.61	31.27	22.01
GRSMC G 053.14 + 00.04	7.78	2.67	32.06	22.56
GRSMC G 022.44 + 00.34	7.83	2.79	32.78	23.08
GRSMC G 049.39 − 00.26	7.90	2.81	35.64	25.09
GRSMC G 019.39 − 00.01	7.99	2.87	35.84	25.23
GRSMC G 034.74 − 00.66	8.27	3.04	36.94	26.00
GRSMC G 023.04 − 00.41	8.28	3.06	38.22	26.90
GRSMC G 018.69 − 00.06	8.30	3.62	40.34	28.40
GRSMC G 023.24 − 00.36	8.57	3.75	41.10	28.93
GRSMC G 019.89 − 00.56	8.64	3.87	41.82	29.44
GRSMC G 022.04 + 00.19	8.69	4.41	47.02	33.10
GRSMC G 018.89 − 00.66	8.79	4.46	47.73	33.60
GRSMC G 023.34 − 00.21	8.87	4.99	48.98	34.48
GRSMC G 034.99 + 00.34	8.90	5.74	50.44	35.50
GRSMC G 029.64 − 00.61	8.90	6.14	55.41	39.00
GRSMC G 018.94 − 00.26	9.16	6.16	55.64	39.16
GRSMC G 024.94 − 00.16	9.17	6.93	56.81	39.99
GRSMC G 025.19 − 00.26	9.72	7.11	58.21	40.97
GRSMC G 019.84 − 00.41	9.97	11.3	58.52	41.19

Discussion and Conclusions

★ Here we have analyzed the Jeans instability mechanism, adopted for star formation, considering the Newtonian approximation of f(R) gravity

★ The related Boltzmann-Vlasov system leads to modified Poisson equations depending on the f(R) model

★ In particular, it is possible to get a new dispersion relation where instability criterion results modified

★ The leading parameter is α, i.e. the second derivative of the specific f(R) model. Standard Newtonian Jeans instability is immediately recovered for $\alpha=0$ corresponding to the Hilbert-Einstein Lagrangian of GR.

★ A new condition for the gravitational instability is derived, showing unstable modes with faster growth rates.

Discussion and Conclusions

- Finally we can observe the instability decrease in f(R)- gravity: such decrease is related to a larger Jeans length and then to a lower Jeans mass
- We have also compared the behavior with the temperature of the Jeans mass for various types of interstellar molecular clouds
- In our model the limit (in unit of mass) to start the collapse of an interstellar cloud is lower than the classical one advantaging the structure formation.
- Real solutions for the Jean mass can be achieved only for $\alpha < 0$ and this result is in agreement with cosmology
- In particular, the condition $\alpha<0$ is essentials to have a well formulated and well-posed Cauchy problem in f(R)- gravity
- Finally, it is worth noticing that the Newtonian value is an upper limit for the Jean mass coinciding with f(R.) = R

Discussion and Conclusions

- It is important to stress that we fully recover the standard collapse mechanisms but we could also describe proto-stellar systems that escape the standard collapse model

- On the other hand, this is the first step to study star formation in alternative theories of gravity

Next Steps

- From an observational point of view, reliable constraints can be achieved from a careful analysis of the proto-stellar phase taking into account magnetic fields, turbulence and collisions

- Addressing stellar systems by this approach could be an extremely important to test observationally f(R) gravity

- Moreover, the approach developed in this work admits direct generalizations for other modified gravities, like nonlocal gravity, modified Gauss-Bonnet theory, string inspired gravity, etc.

- Developing further this approach gives, in general, the possibility to confront the observable dynamics of astrophysical objects (like stars) with predictions of alternative gravities.